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Analytic Static Deflection Solutions of Beams Resting on Strong Nonlinear Elastic Foundations

Chao-Kuang Chen **, Li-Kuo Chou* and Sen-Yung Lee**

Keywords： deflection of beams, strong nonlinear elastic foundation, Modified Adomian method, perturbation method

Abstract

The analytic static deflection solutions of beams resting on nonlinear elastic foundations are developed by the modified Adomian method. If the applied force function is an analytic function, then the deflection function can be derived and expressed in Maclaurin series. A recurrence relation for the coefficients of the Maclaurin series is derived. It is shown that the proposed solution method is accurate and efficient. The solution method can be successfully applied to the problem with strong nonlinearity. The results are also compared with those obtained by the perturbation method. It is found that the error of the perturbation solution will increase not only when the nonlinear parameter is increased but also when the applied load is increased.

Introduction

Beams are basic structures and widely used in engineering application. The problem of beams on linear elastic foundation has been studied by many investigators (Hetenyi, 1946; Lee et al 1991, 1992). When the deformation of beams is large, the nonlinear analysis turns to be important.

In addition to material and geometric nonlinearities, the nonlinear elastic foundation is also a source of nonlinearities coming into play in structural mechanics. The static deflection of a uniform beam resting on a nonlinear elastic foundation has been studied by many investigators.

Distefano and Todeschini (1975) studied the solution of a beam on nonlinear elastic foundation by applying the quasi-linearization approach. Sharma and DasGupta (1975) investigated the bending deflection of axially constrained beams on nonlinear Winkler elastic foundation by an iteration method using Green’s functions. Beaufait and Hoadley (1980) used a bi-linear curve to approximate the load-deflection relationship of a nonlinear foundation. Yankelevsky *et al*. (1989) used an iterative procedure based on the exact stiffness matrix for beam on Winkler foundations. Sayegh and Tso (1992) developed a numerically integrated finite element to solve the bending deflection of a curved elastic beam supported by a nonlinear foundation. Using the method of perturbation, the static deflection of non-uniform beams resting on nonlinear elastic foundation was investigated by Kuo and Lee (1994). Tsiatas (2010) presented a boundary integral equation solution to the non-linear deflection of beams resting on a nonlinear tri-parametric elastic foundation. The interval of the beam was divided into N equal sub-elements.

From the existing literatures, it can be found that many of the solution methods will accumulate error (Distefano and Todeschini, 1975; Sharma and DasGupta, 1975; Beaufait and Hoadley, 1980; Yankelevsky *et al*. 1989; Tsiatas, 2010) or be valid only for the problem with small nonlinearity (Kuo and Lee, 1994) or require tedious calculation(Yankelevsky *et al*. 1989, Sayegh and Tso 1992; Tsiatas ,2010). A simple and straight closed form series solution for the analytic static deflection of a beam resting on a nonlinear elastic foundation is not available. In this paper, the modified Adomian method (Adomian, 1994) is applied to develop the closed form solution for the static deflection of beams resting on nonlinear elastic foundations. The applied force function is assumed to be an analytic function. The non-dimensional length of the beam is set to be one such that the convergence of non-dimensional deflection function, expressed in a Maclaurin series, will be ensured. The proposed solution method is shown to be simple, efficient and can be successfully

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* Professors, Department of Mechanical Engineering, National Cheng Kung University, Tainan 701, Taiwan, R.O.C.

** Graduate student, Department of Mechanical Engineering, National Cheng Kung University, Tainan 701, Taiwan, R.O.C.

applied to the problem with strong nonlinearity. The results are also compared with those obtained by the perturbation method.

Governing Equation and Boundary Conditions

Consider the static deflection of a cantilever Bernoulli-Euler beam resting on non-linear elastic foundation, as shown in Figure 1. Based on the Bernoulli-Euler beam theory, in terms of the following non-dimensional quantities,

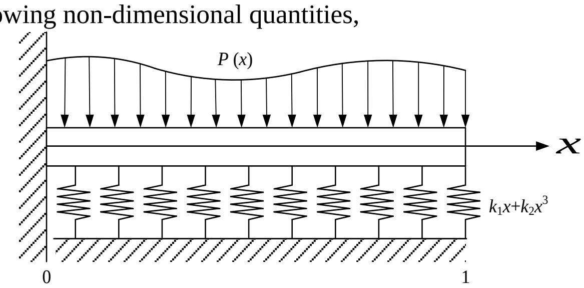


Fig. 1: Geometry and coordinate system of a cantilever beam resting on nonlinear elastic foundation subjected to distributed load.

$$x=\frac{X}{l},\qquad V(x)=\frac{\bar{V}(X)}{l},\qquad P(x)=\frac{p(X)l^3}{E(0)I(0)},$$

$$k_1=\frac{K_1 l^3}{E(0)I(0)},\qquad k_2=\frac{K_2 l^3}{E(1)I(1)},\tag{1}$$

the governing differential equation of the system is

$$\frac{d^4V(x)}{dx^4}+k_1V(x)+k_2V^3(x)=P(x),\ x\in(0,1),\tag{2}$$

and the associated boundary conditions are

at $x = 0$:

$$V(0)=0,\tag{3}$$

$$\frac{dV(0)}{dx}=0,\tag{4}$$

at $x = 1$:

$$\frac{d^2V(1)}{dx^2}=0,\tag{5}$$

$$\frac{d^3V(1)}{dx^3}=0,\tag{6}$$

where $\bar{V}(X)$ is the flexural displacement, X is the coordinate along the beam, and l is the beam length. E, I, K_1, K_2 and $p(x)$ denote the Young's modulus, the area moment of inertia, the linear and the nonlinear elastic foundation modulus and the applied transverse forces per unit length, respectively.

Modified Adomian Decomposition Method

If the applied distributed load function is an analytic function, one can assume the deflection function $V(x)$ and the force function $P(x)$ to be in the following Maclaurin series forms

$$V(x)=\sum_{m=0}^{\infty}a_m x^m,\tag{7}$$

$$P(x)=\sum_{m=0}^{\infty}P_m x^m,\tag{8}$$

Consequently,

$$\frac{dV}{dx}=\sum_{m=0}^{\infty}(m+1)a_{m+1}x^m,\tag{9}$$

$$\frac{d^2V}{dx^2}=\sum_{m=0}^{\infty}(m+1)(m+2)a_{m+2}x^m,\tag{10}$$

$$\frac{d^3V}{dx^3}=\sum_{m=0}^{\infty}(m+1)(m+2)(m+3)a_{m+3}x^m,\tag{11}$$

$$\frac{d^4V}{dV^4}=\sum_{m=0}^{\infty}(m+1)(m+2)(m+3)(m+4)a_{m+4}x^m\tag{12}$$

The nonlinear term $V^3(x)$ can be expressed in the following form

$$\begin{aligned}V^3(x)&=\left\{\sum_{n=0}^{\infty}a_n x^n\right\}\left\{\sum_{v=0}^{\infty}a_v x^v\right\}\left\{\sum_{\mu=0}^{\infty}a_\mu x^\mu\right\}\\&=\sum_{m=0}^{\infty}A_m(a_0,\ldots,a_m)x^m,\end{aligned}\tag{13}$$

where the coefficients of Adomian polynomials (Adomian, 1994) are

$$A_m(a_0,\ldots,a_m)=\sum_{v=0}^{m}\sum_{\mu=0}^{v}a_{m-v}a_{v-\mu}a_\mu\tag{14}$$

For convenience, some coefficients of Adomian polynomials for the nonlinearity are listed

$$A_0=a_0^3\tag{15}$$

$$A_1=3a_0^2a_1\tag{16}$$

$$A_2=3a_0^2a_2+3a_1^2\tag{17}$$

$$A_3=3a_1^3+3a_0^2a_3+6a_0a_1a_2\tag{18}$$

$$A_4=3a_0^2a_4+3a_1^2a_2+3a_2^2a_0+6a_0a_1a_3\tag{19}$$

$$\begin{aligned}A_5&=3a_0^2a_5+3a_1^2a_3+3a_2^2a_1+6a_0a_1a_4\\&\quad+6a_0a_2a_3\end{aligned}\tag{20}$$

$$\begin{aligned}A_6&=a_2^3+3a_0^2a_6+3a_1^2a_4+3a_3^2a_0+6a_0a_1a_5\\&\quad+6a_0a_2a_4+6a_1a_2a_3\end{aligned}\tag{21}$$

$$\begin{aligned}A_7&=3a_0^2a_7+3a_1^2a_5+3a_2^2a_3+3a_3^2a_1+6a_0a_1a_6\\&\quad+6a_0a_2a_5+6a_0a_3a_4+6a_1a_2a_4\end{aligned}\tag{22}$$

$$\begin{aligned}A_8&=3a_0^2a_8+3a_1^2a_6+3a_2^2a_4+3a_3^2a_2+3a_4^2a_0\\&\quad+6a_0a_1a_7+6a_0a_2a_6+6a_0a_3a_5+6a_1a_2a_5\\&\quad+6a_1a_3a_4\end{aligned}\tag{23}$$

$$\begin{aligned}A_9&=a_3^3+3a_0^2a_9+3a_1^2a_7+3a_2^2a_5+3a_4^2a_1\\&\quad+6a_0a_1a_8+6a_0a_2a_7+6a_0a_3a_6\\&\quad+6a_0a_4a_5+6a_1a_2a_6+6a_1a_3a_5+6a_2a_3a_4,\end{aligned}\tag{24}$$

$$\begin{aligned}A_{10}&=3a_0^2a_{10}+3a_1^2a_8+3a_2^2a_6+3a_3^2a_4\\&\quad+3a_4^2a_2+3a_5^2a_0+6a_0a_1a_9+6a_0a_2a_8\\&\quad+6a_0a_3a_7+6a_0a_4a_6+6a_1a_2a_7\\&\quad+6a_1a_3a_6+6a_1a_4a_5+6a_2a_3a_5,\end{aligned}\tag{25}$$

$$\vdots$$

Substituting equations (7-13) back to equation (2), one has

$$\sum_{m=0}^{\infty}(m+1)(m+2)(m+3)(m+4)a_{m+4}x^m + k_1\sum_{m=0}^{\infty}a_m x^m + k_2\sum_{m=0}^{\infty}A_m(a_0,...,a_m)x^m = \sum_{m=0}^{\infty}P_m x^m \quad (26)$$

After collecting coefficients of like power, the following recurrence relation can be obtained

$$a_{m+4} = P_m - k_1 a_m - k_2 A_m \,/(m+1)(m+2)(m+3)(m+4) \quad (27)$$

From this recurrence relation and equation (14), one observes that all the coefficients a_m can be expressed in terms of four coefficients, (a_0, a_1, a_2, a_3). These four coefficients will be determined from the specified four boundary conditions, equations (3-6). Consequently, all the coefficients a_m of the Maclaurin series for the deflection function $V(x)$ is determined.

In numerical analysis, only finite terms in polynomial are employed to approximate the solution. When n+1 terms are used, the approximated deflection function is

$$V(x) = \sum_{m=0}^{n} a_m x^m, \quad (28)$$

and the coefficients of Adomian polynomials are

$$A_n(a_0,...,a_n) = \sum_{v=0}^{n}\sum_{\mu=0}^{v} a_{n-v}a_{v-\mu}a_{\mu} \quad (29)$$

For the problem studied, the four boundary conditions, equations (3-6), are reduced to

$$V(0) = a_0 = 0 \quad (30)$$

$$\frac{dV(0)}{dx} = a_1 = 0 \quad (31)$$

$$\frac{d^2V(1)}{dx^2} = \sum_{m=0}^{n}(m+1)(m+2)a_{m+2} = 0 \quad (32)$$

$$\frac{d^3V(1)}{dx^3} = \sum_{m=0}^{n}(m+1)(m+2)(m+3)a_{m+3} = 0 \quad (33)$$

The coefficients a_4 ~ a_n can be simplified and expressed in terms of a_2 and a_3,

$$a_4 = P_0/24 \quad (34)$$

$$a_5 = P_1/120 \quad (35)$$

$$a_6 = P_2 - k_1 a_2/360 \quad (36)$$

$$a_7 = P_3 - k_1 a_3/840 \quad (37)$$

$$a_8 = P_4 - k_1 a_4/1680 \quad (38)$$

$$a_9 = P_5 - k_1 a_5/3024 = P_5 - k_1 P_1/(3024\times 120) \quad (39)$$

$$\begin{aligned} a_{10} &= P_6 - k_1 a_6 - k_2 a_2^3/5040 \\ &= P_6 - k_1(P_2 - k_1 a_2/360) - k_2 a_2^3/5040 \end{aligned} \quad (40)$$

$$\vdots$$

$$a_n$$

These two coefficients, a_2 and a_3, will be determined from equations (32-33).

Verification and Examples

To verify the previous analysis, two examples are illustrated.

Example 1: Consider the problem with the same non-dimensional governing equation and the associated boundary conditions as given in equations (2-6). Two non-dimensional spring constants are k_1 = k_2 = 1. The non-dimensional applied load is given in the following polynomial form

$$\begin{aligned} P(x) = {} & x^{12} - 12x^{11} + 66x^{10} - 208x^9 + 396x^8 \\ & - 432x^7 + 216x^6 + x^4 - 4x^3 + 6x^2 + 24. \end{aligned} \quad (41)$$

When 16 terms are used to approximate the deflection function, N = 15 in equation (28). Following the solution method as revealed, the 16 coefficients, a_0 ~ a_{15}, satisfy the following algebra equations:

$$a_0 = V(0) = 0 \quad (42)$$

$$a_1 = V^{(1)}(0) = 0 \quad (43)$$

$$\sum_{m=0}^{N}(m+1)(m+2)a_{m+2} = V^{(2)}(1) = 0 \quad (44)$$

$$\sum_{m=0}^{N}(m+1)(m+2)(m+3)a_{m+3} = V^{(3)}(1) = 0 \quad (45)$$

$$a_4 = 1 \quad (46)$$

$$a_5 = 0 \quad (47)$$

$$a_6 = 6 - a_2/360 \quad (48)$$

$$a_7 = -4 - a_3/840 \quad (49)$$

$$a_8 = 1 - a_4/1680 \quad (50)$$

$$a_9 = -a_5/3024 \quad (51)$$

$$a_{10} = 216 - a_6 - a_2^3/5040 \quad (52)$$

$$a_{11} = 432 - a_7 - 3a_3^2 a_1/7920 \quad (53)$$

$$a_{12} = 396 - a_8 - (3a_2^2 a_4 + 3a_3^2 a_2)/11880 \quad (54)$$

$$a_{13} = -208 - a_9 - (a_3^3 + 3a_2^2 a_5 + 6a_2 a_3 a_4) \,/17160 \quad (55)$$

$$\begin{aligned} a_{14} = {} & 66 - a_{10} - (3a_2^2 a_6 + 3a_3^2 a_4 + 3a_4^2 a_2 \\ & + 6a_2 a_3 a_5)/24024 \end{aligned} \quad (56)$$

$$\begin{aligned} a_{15} = {} & -12 - a_{11} - (3a_3 a_4^2 + 3a_2^2 a_7 + 3a_3^2 a_5 \\ & + 6a_2 a_3 a_5 + 6a_2 a_4 a_5)/32760 \end{aligned} \quad (57)$$

As a result, these coefficients can be determined as

$a_2 = 6,\ a_3 = -4,\ a_4 = 1$ and

$$a_0 = a_1 = \cdots = a_{15} = 0 \quad (58)$$

After substituting these obtained coefficients back into Equation (7), the exact solution of the system is obtained.

$$V(x) = x^6 - 4x^4 + 6x^2. \quad (59)$$

Example 2: To illustrate the convergence of proposed method and compare the solutions with those obtained by the perturbation method, one consider the deflection of a cantilevered uniform steel beam with square cross section, resting on a non-linear elastic foundation subjected to uniform distributed force. The material, the geometric properties and the applied force are given as:

$E = 2 \cdot 10^8\ KN/m^2$,

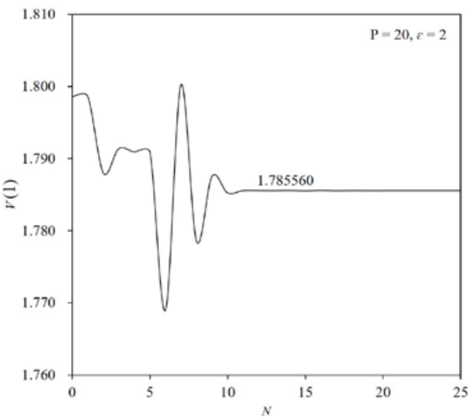
I = 250 · 10^{-8} m^4 (width = height = 7.4 cm),
K_1 = 500 KN/m, (60)
p = 500P KN/m
l = 1 m.

The non-dimensional governing differential equation of the system is

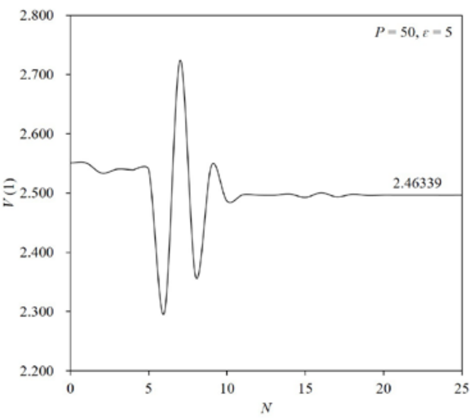
$$\frac{d^4V(x)}{dx^4}+V(x)+\varepsilon V^3(x)=P,\ x\in(0,1), \tag{61}$$

Where ε denotes the ratio between two non-dimensional spring constants, k_2/k_1.

Figures 2(a) and 2(b) show the convergence of the solutions with different ε and applied load P, evaluated by the proposed solution method. It can be found that even the nonlinear parameter and the applied loads are large; the solutions converge fast and converge to steady values when the number of term employed is less than 20. It also shows that the proposed solution method is efficient and can be successfully applied to the problem with strong nonlinearity. In the following numerical analysis, 25 terms are used. The convergence of the solution is ensured.



(2a)



(2b)

Fig. 2: Convergence of the solutions with different ε and applied load P, evaluated by the proposed solution method

To compare the deflections of the beams, with different nonlinear parameter ε and applied load P, evaluated by the proposed solution method with those obtained by the perturbation method, figures 3-6 and one table are present.

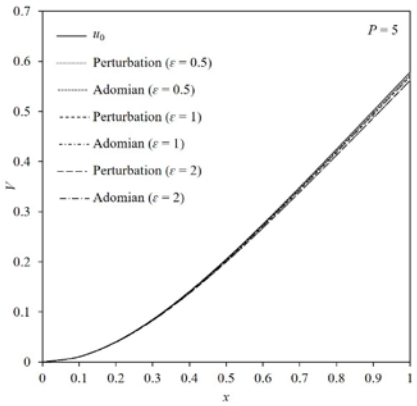


Fig. 3: Deflection of the beam evaluated by the proposed method and the perturbation method with P = 5 and various nonlinear parameter ε.

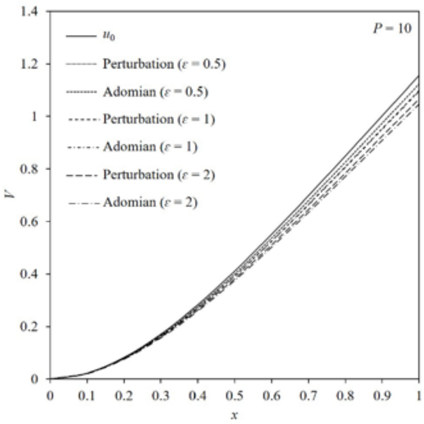


Fig. 4: Deflection of the beam evaluated by the proposed method and the perturbation method with P = 10 and various nonlinear parameter ε.

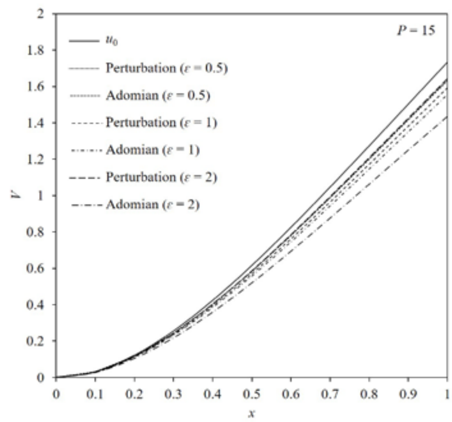


Fig 5. Deflection of the beam evaluated by the

proposed method and the perturbation method with P = 15 and various nonlinear parameter ε.

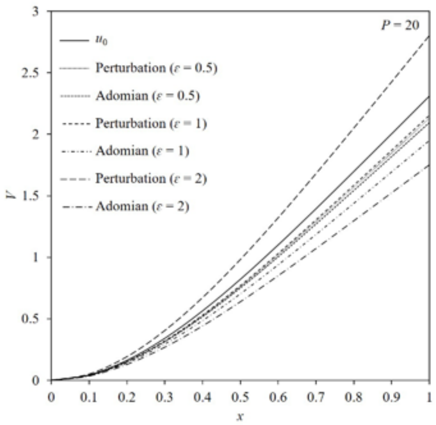


Fig. 6: Deflection of the beam evaluated by the proposed method and the perturbation method with P = 20 and various nonlinear parameter ε.

Table: Static deflection of cantilevered beams resting nonlinear elastic foundation subjected to distribution load [A: Adomian method, B: Perturbation method]

		$P = 5$		$P = 10$		$P = 15$		$P = 20$	
		A	B	A	B	A	B	A	B
$x = 0.5$	$\varepsilon = 0$	0.205	0.205	0.410	0.410	0.616	0.616	0.821	0.861
	$\varepsilon = 0.5$	0.204	0.204	0.400	0.400	0.586	0.585	0.750	0.762
	$\varepsilon = 1$	0.202	0.203	0.391	0.392	0.557	0.569	0.703	0.769
	$\varepsilon = 2$	0.200	0.200	0.375	0.381	0.520	0.586	0.639	0.982
$x = 1$	$\varepsilon = 0$	0.578	0.578	1.155	1.155	1.733	1.733	2.310	2.310
	$\varepsilon = 0.5$	0.573	0.573	1.123	1.123	1.632	1.638	2.094	2.131
	$\varepsilon = 1$	0.569	0.569	1.094	1.094	1.632	1.632	1.949	2.152
	$\varepsilon = 2$	0.561	0.562	1.047	1.065	1.434	1.642	1.752	2.802

In the analysis by the perturbation method (Nayfeh, 1993), via straight forward expression, the solution is in the form of

$$V(x) = u_0 + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + \cdots \tag{62}$$

Substituting this solution into equation (61) and collecting coefficients of like power of ε, one has the following linear differential equations with homogenous boundary conditions:

coefficient of ε^0:

$$\frac{d^4 u_0(x)}{dx^4} + u_0(x) = P(x) \tag{63}$$

coefficient of ε^1:

$$\frac{d^4 u_1(x)}{dx^4} + u_1(x) + u_0^3 = 0 \tag{64}$$

coefficient of ε^2:

$$\frac{d^4 u_2(x)}{dx^4} + u_2(x) + 3u_0^2 u_1 = 0 \tag{65}$$

N

The associated boundary conditions for u_i, i = 0, 1, 2 are

at x = 0:

$$u_i(0) = 0, \tag{66}$$

$$\frac{du_i(0)}{dx} = 0, \tag{67}$$

at x = 1:

$$\frac{d^2 u_i(1)}{dx^2} = 0, \tag{68}$$

$$\frac{d^3 u_i(1)}{dx^3} = 0, \tag{69}$$

After having these three solutions u_i, i = 0, 1, 2, (Hetenyi, 1946; Lee et al 1991, 1992), the solution is obtained by restoring them back to equation (62).

Figures 3-6 and a table illustrate the static deformation of the beam with various nonlinear parameters ε and applied loads. Solutions evaluated by the proposed solution method and the perturbation method are listed and compared. It can found that when the nonlinear parameter is zero, the static deflections evaluated by two different methods are the same. One also has the following observations:

(1) All the nonlinear static deflections ($\varepsilon > 0$) evaluated by the proposed method are less than those obtained by the perturbation method.
(2) All the nonlinear static deflections evaluated by the proposed method will decrease as the nonlinear parameter ε is increased. However, this is not the case for the perturbation solutions with large nonlinear parameter ε and applied load P.
(3) At the same location, with the same nonlinear parameter ε, the error of the perturbation solution will increase as the applied load P is increased.
(4) With the same applied load, the error of the perturbation solution will increase as the nonlinear parameter ε is increased.
(5) The error of the perturbation solution will increase not only when the nonlinear parameter is increased but also when the applied load is increased.
(6) When the applied load is small, say P = 5, even in the case that the nonlinear parameter ε is greater than 1, the difference between the deflections evaluated by two different method is small.

Conclusions

In this paper, the modified Adomian method is successfully applied to develop the closed form series solution for the static deflection of a beam resting on strong nonlinear elastic foundation. The proposed solution method is shown to be accurate and efficient and can be successfully applied to the problem with strong nonlinearity. The results are also compared with those obtained by the perturbation method. It is found that the error of the perturbation solution will increase not only when the nonlinear parameter is increased but also when the applied load is increased. The proposed solution method can also be extended and applied to the non-uniform beam problems and the problems with different boundary conditions.

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Nomenclature

X	coordinate along the beam
x	space variable along the beam
E	Young’s modulus
I	area moment of inertia
l	beam length
K_1	linear elastic foundation modulus
K_2	nonlinear elastic foundation modulus
$\bar{V}(X)$	flexural displacement
$V(x)$	deflection function
$P(\mathrm{x})$	force function
$p(\mathrm{x})$	transverse force
k_1	non-dimensional linear spiral
k_2	non-dimensional nonlinear spiral
$\varepsilon = k_1 + k_2$	the ratio between two non-dimensional sping constant

樑在強非線性彈性基底上時之靜態撓曲的解析解

陳朝光、周立國、李森墉
國立成功大學機械工程學系

摘要

本文利用Adomian方法來求取樑在強非線性彈性基底上時之靜態撓曲的解析解，如果外力之函數式可析函數(Analysis function)，則推導所得的樑之非線性撓曲可以馬克勞林級數表示之，同時亦推導出此馬克勞林級數之係數間的遞迴關係式。結果顯示，所提出的方法是精確有效的，有效的成功應用於強非線性問題，結果獲得與擾動法(Perturbation)做比較。發現擾動法解的誤差將隨之非線性係數之增加而增加，同時亦將隨著外力的增加而增加。