Dynamic Characteristic Analysis of a Two-Stage Water Rocket in Flight

Feng-Yao Chang*and Huei Chu Weng**

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ABSTRACT

This study explored the flight trajectory and dynamic characteristics of a two-stage water rocket theoretically. The main purpose is to analyze the influences of the air pressure and water volume in the first and second stages of the water rocket on the vertical flight height, velocity, and acceleration of the second stage (upper part of the water rocket) over time. During flight, the pressure difference caused by a one-way check valve between the first stage and the second stage is a key factor determining the separation of the two stages. This flight modeling is based on the water rocket's momentum conservation equation, the outlet fluid velocity condition, and the relationship between internal air pressure and water volume. Through a numerical solution for the problem, the vertical flight height, velocity and acceleration of the water rocket were predicted. These predictions were compared with experimental data in advance. It was found that results in predictions are close to the experimental data, and the check valve in the two-stage water rocket can effectively make the rocket's vertical flight height higher than that of the single-stage water rocket. The larger the check valve pressure difference that the one-way check valve can withstand, the higher the flight height of the second stage after being detached from the first stage. Under a certain air pressure and total water volume in the water rocket, the best flight performance can be achieved by adjusting the water volume ratio between the upper and lower parts of the water rocket.

INTRODUCTION

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- * Ph.D. Candidate, Department of Mechanical Engineering, Chung Yuan Christian University, Taoyuan 32023, Taiwan, ROC.
- ** Professor, Department of Mechanical Engineering, Chung Yuan Christian University, Taoyuan 32023, Taiwan, ROC.

The flight of this rocket involves a number of complex influencing factors, including the air and water inside the rocket, the external drag caused by the shape design of the rocket, and the thermal-flow state of its fuel. The theory and model developed by this research could be further applied to many different aerospace, military, bionic, and robotic applications (Liu et al., 2019; Mo et al., 2019; Zufferey et al., 2019).

Regarding the research on the water rocket's propulsion process, Watanabe et al. (2003) used high-speed photography to observe the water level change of a launched water rocket, showing that after the water rocket was launched, the water was pushed out only 350 ms. Then, due to the mixing of air and water, the air-water mixture started to be discharged. The water in the rocket moved in a rotating state until it was completely exhausted. Because the physical phenomena in the air-water mixing stage are very complicated, it is difficult to analyze with simple theory. Fu (2004) established a propulsion model where air and water are discharged separately, that is, the water is discharged first and then the air is discharged. After comparing with experimental results, it was found that this model can get quite good predictions. Barrio-Perotti et al. (2010) also believed that during the water rocket's propulsion process, air discharge cannot be ignored. Therefore, they proposed to use the model of discharging water and then air for theoretical analysis of a water rocket in flight. In order to enhance the performance of a water rocket to achieve the maximum possible burnout velocity, Al-Qutub et al. (2014) used an improved rocket nozzle and adjusted the water to air mass ratio in the water rocket for an energy balance analysis. The results showed that when the mass ratio is about 7, the performance of the water rocket is the optimum.

Regarding the research on the external drag of a water rocket during its flight, the shape design of the rocket will determine the drag coefficient, and the setting value of this coefficient will also determine whether the predictions of the flight trajectory and dynamic characteristics of the rocket are accurate (Fischer et al., 2019). Watanabe et al. (2004) measured the drag coefficients of a water rocket with a concave hemispherical nose and a water rocket with

a no concave hemispherical nose through a wind tunnel test. The results showed that the value in the case with a concave hemispherical nose is relatively low. Although wind tunnel tests can obtain relatively accurate drag coefficients for water rockets, their costs are relatively high. Barrio-Perotti et al. (2009) proposed a simple free-fall test method to measure the drag coefficient of a water rocket. The method is to measure the time it takes for the rocket to fall from a height of 14 m to the ground., and then use the second-order ordinary differential equation constructed by Newton's second law to solve the drag through computer rocket's coefficient calculation. This method is relatively simple, and its cost is relatively low. The drag coefficient deviation obtained by this method is about 3%. One can further explore how the experimental design affects the drag coefficient for a falling object (Siebert et al., 2019).

In the construction of water rocket's momentum conservation equation, the Reynolds transport theorem is traditionally used to analyze the propulsion performance of a water rocket. Lee et al. (2001) applied the Lagrangian Reynolds transport equation to derive the momentum conservation equation. Prusa (2000) and Barrio-Perotti et al. (2010) argued that within an appropriate control volume, the Reynolds transport theorem can be used to construct the momentum conservation equation for water rockets. As for the relationship between the air and water in a water rocket, under the assumption of ignoring the fluid viscosity effect, Shia (1998), Chiou (1999), Lee et al. (2001), and Fu (2004) applied the continuity equation and the Bernoulli equation to calculate the fluid flow velocity at the outlet, and derived the change in the amount of air and water. During the flight of a water rocket, due to the rapid expansion of the high-pressure fluid inside the rocket in a short time, Shia (1998), Prusa (2000), and Fu (2004) regarded the expansion process as an isentropic process. Using the above-mentioned theories or assumptions, one can numerically predict the flight trajectory and dynamic characteristics of a water rocket.

Stage separation is an important flight operation for a multi-stage missile, aircraft, or spacecraft. Its purpose is to reduce unnecessary weight load to increase the flight speed and distance of the flying object. Jevakumar et al. (2005) used equations of motion to perform a stage separation dynamic analysis on the multi-stage aircraft of a retro rocket system, so as to calculate the relationship among the separation time, relative velocity, lateral gap, and longitudinal clearance between separated objects. The statistical method used in this study was also applied to examine the effect of design parameters on the separated objects. With reference to the fluid characteristics in the gas resonance tube, Gao and Fu (2012) conducted an aerodynamic-dynamic coupled simulation on the stage separation process of a

multi-stage rocket, and analyzed the flow and dynamic characteristics between stages. The results of this study showed that during the stage separation process, the fluid flow field between stages presents five-period changes, namely the initial transient period, the smooth gas leakage period, the gas resonance period, the gas resonance suspension period, and the gas resonance recovery period. Jamilnia et al. (2012) dealt with the trajectory optimization and stage separation optimization of multi-stage launch vehicles using two different approaches, namely separated approach and integrated approach. This study obtained the simultaneous optimization of staging and trajectory of multi-stage launch vehicles, and found that the two approaches can obtain similar solutions. Samani and Pourtakdoust (2014) performed the simulation on the stage separation process of a two-stage supersonic parachute projectile and determined the relative safety distance between stages at a certain time after the disconnection has been completed. This study also found that the application of the Monte Carlo statistical method is effective for the analysis of the uncertainty of the separation system and can provide useful knowledge for the initial design stage of the separation system. Mulsow (2011) devoted their efforts to the simulation of the optimization of a two-stage water rocket by using Monte Carlo method. The results showed that to achieve the maximum flight height of the rocket, the volume ratios of water to air in the second stage (upper part of the water rocket) and the first stage (lower part of the water rocket) are 0.3423 and 0.4577, respectively.

Until now, most studies on water rockets have focused mainly on a single stage as described above. The two-stage water rocket problem that has been discussed in the literature only studied the optimization simulation; therefore, there were no studies on flight trajectory, dynamic characteristics, and stage separation of two-stage water rockets. In this study, a one-way check valve was designed and combined with two single-stage water rockets to construct a two-stage water rocket system. The flight modeling was then carried out based on the water rocket's momentum conservation equation, the propulsion mode, the outlet fluid velocity condition, and the relationship between internal air pressure and water volume. The numerical maximum flight heights of the second stage obtained for different drag coefficients were compared with the experimental data to verify the validity of the flight model. Based on this flight mode, the effects of the check valve pressure difference, air pressure inside the rocket, total water volume inside the rocket, and first-stage water volume fraction on the flying height, velocity, and acceleration of the two-stage water rocket were further analyzed.



Fig. 1. Schematic diagram of forces acting on the two-stage water rocket discussed in this study.

MATHEMATICAL MODE

Operation of a Two-Stage Water Rocket

Figure 1 shows the two-stage water rocket used in this study. This rocket was designed according to the two-stage water rocket patent of Joffe (1962) and can be divided into two stages: the first stage (lower part of the rocket) is a two-chamber structure, and the second stage (upper part of the rocket) is a single-chamber structure. There is a one-way check valve used to connect the two stages of the rocket filled with water. This valve is mainly composed of two concentric tubes of similar size. The inner tube of harder material is combined with the second stage, and the outer tube of softer material is combined with the first stage. Through the friction between the outer wall of the inner tube and the inner wall of the outer tube, this valve can suppress the separation of the two stages. When the air is pumped and pressurized into the first stage, the air can enter the second stage through the check valve but cannot flow back to the first stage due to the blockage of the ball in the valve. After the launch of the water rocket, the water or air in the first stage is discharged, and the air pressure in the first stage decreases accordingly. At the same time, the outer tube of the check valve expands to maintain the force balance, resulting in less friction with the inner tube. Once the friction can no longer suppress the stage separation, the second stage separates from the first stage and flies alone. It can be seen from the operating principle of the one-way check valve mentioned above that the flight process of a two-stage water rocket can be regarded as a two flight processes of a single-stage water rocket. Therefore, to obtain the flight trajectory and dynamic characteristics of a two-stage water rocket, the flight modeling of a single-stage water rocket must be applied.

Flight Modeling of a Single-Stage Water Rocket

In this study, the traditional Reynolds transport theorem is used to derive the momentum conservation equation for a single-stage water rocket, and the propulsion mode of the water rocket is built based on a process where air and water are discharged separately. The flight modeling of a single-stage water rocket can then be carried out by means of the combination of the momentum conservation equation, the propulsion mode, and the outlet fluid velocity condition as well as the relationship between internal air pressure and water volume.

(1) Momentum Conservation Equation

If the main body of the water rocket is regarded as the system, the volume in the system can be regarded as the control volume and the main body surface can be regarded as the control surface. According to the Reynolds transport theorem, the momentum change rate of the water rocket system resulted from the fluid flow can be divided into two integral terms, namely, the momentum change rate of the control volume and the net momentum change rate of the control surface:

$$\frac{\partial}{\partial t} \int_{cv} u_z \rho d\Psi + \int_{cs} u_z \rho \bar{u} \cdot \hat{n} dA \tag{1}$$

where t is time, \vec{u} is the vector of velocity, u_z is the velocity of the fluid in z direction, ρ is the fluid density, A is the surface area of system, \forall is the volume of system, and \hat{n} is the unit normal vector on control surface. Therefore, the momentum change rate in the flying water rocket system D(MV)/Dt is

$$\frac{D(MV)}{Dt} = \frac{\partial}{\partial t} \int_{cv} u_z \rho d\Psi + \int_{cs} u_z \rho \vec{u} \cdot \hat{n} dA + M(t) \dot{V}_{cv} \quad (2)$$

According to Newton's second law of motion, all external forces acting on the flying water rocket system are the momentum change rate in the flying water rocket system. While there are three external forces acting on the flying water rocket, they are air drag (D), gravity (G), and outlet pressure (F_{pe}). Thus,

$$\sum F_{z} = \frac{\partial}{\partial t} \int_{cv} u_{z} \rho d\Psi + \int_{cs} u_{z} \rho \bar{u} \cdot \hat{n} dA + M(t) \dot{V}_{cv} \quad (3)$$

$$\sum F_{z} = D + G + F_{pe}$$

= $-\frac{1}{2} \rho_{air} V_{cv}^{2} A_{cs} C_{D} - M(t)g + P_{e} A_{e}$ (4)

where ρ_{air} is the air density, V_{cv} is the flight velocity of the water rocket, A_{cs} is the external cross-sectional area of the water rocket, C_D is drag coefficient, M(t) is the total mass of the water rocket, g is gravity acceleration, P_e is the outlet pressure of the water rocket, A_e is the outlet cross-sectional area of the water rocket. Furthermore, the momentum change rate term on the control surface can be derived as

$$\int_{cs} u_z \rho \vec{u} \cdot \hat{n} dA = -v_e \rho (+v_e) A_e = -\rho v_e^2 A_e$$
(5)

Then, the momentum change rate term on the control volume can be also derived as

$$\frac{\partial}{\partial t} \int_{cv} u_z \rho d\Psi = \frac{\partial}{\partial t} (u_z m_i) = \dot{u}_z m_i + u_z \dot{m}_i \tag{6}$$

where v_e is the relative average velocity of the fluid at the outlet, m_i is the fluid mass in the control volume, \dot{m}_i is the fluid mass flow rate in the control volume, and \dot{u}_z is the fluid acceleration in the *z* direction.

The term $\dot{u}_z m_i$ in Eq. (6) can be further stated as

$$\dot{u}_z m_i = -m_i \frac{A_e}{A_i} \dot{v}_e \tag{7}$$

The other item $u_z \dot{m}_i$ can also be stated as

$$u_z \dot{m}_i = \dot{m}_e \frac{A_e}{A_i} v_e \tag{8}$$

Here $\dot{m}_e = \rho v_e A_e$ is the outlet fluid mass flow rate, A_i is the inner cross-sectional area of the water rocket, and \dot{v}_e is the relative average acceleration of the fluid at the outlet.

Through Eqs. (2)–(8), the momentum conservation equation for the water rocket can be derived as

$$-\frac{1}{2}\rho_{air}V_{cv}^{2}A_{cs}C_{D} - M(t)g + P_{e}A_{e}$$

$$= M(t)\dot{V}_{cv} - m_{i}\frac{A_{e}}{A_{i}}\dot{v}_{e} + (\frac{A_{e}}{A_{i}} - 1)\rho A_{e}v_{e}^{2}$$
(9)

(2) Relationship between Outlet Fluid Velocity and Air Pressure within Water Rocket

When the water rocket is launched, the high-pressure air in the water rocket expands rapidly in a short time, and the water in the water rocket is quickly discharged. It is assumed that the viscous effect of water in the water rocket is negligible, and the rapid expansion of the air volume in the water rocket can be considered as an isentropic process under adiabatic expansion conditions. According to the Bernoulli equation, the relationship between the air pressure in the water rocket P_i , the level speed in the rocket v_i , the outlet pressure of the rocket P_e , and the relative average velocity of the fluid at the outlet v_e are as follows:

$$P_i + \frac{1}{2}\rho v_i^2 + \rho g h_i = P_e + \frac{1}{2}\rho v_e^2 + \rho g h_e$$
(10)

where h_i is the level position of the fluid within the water rocket, h_e is the outlet position of the rocket, and g is the gravity acceleration. Then, the relative average velocity of the fluid at the outlet v_e can be obtained by means of the continuity equation $A_iv_i = A_ev_e$ as follows:

$$v_e = \sqrt{\frac{2(P_i - P_e) + 2\rho g(h_i - h_e)}{\rho(1 - A_e^2 / A_i^2)}}$$
(11)

In a certain time interval Δt , the relationship between the relative average velocity of the fluid at the outlet v_e and the relative average acceleration of the fluid at the outlet \dot{v}_e is

$$v_e(t + \Delta t) = v_e(t) + \dot{v}_e(t)\Delta t \tag{12}$$

(3) Relationship between Air Pressure and Air Volume within Water Rocket

Because the water in the water rocket is completely expelled by the high-pressure air for a very short time, the rapid expansion of the air volume in the water rocket can be considered as an isentropic process under adiabatic expansion conditions. Therefore, the relationship between the air pressure within the water rocket P_i and the air volume Ψ_{air} within the rocket in a certain time interval Δt can be expressed as follows:

$$P_i(t) \mathcal{V}_{air}^k(t) = P_i(t + \Delta t) \mathcal{V}_{air}^k(t + \Delta t)$$
(13)

where *k* is the specific heat ratio, which is the ratio of the constant-pressure specific heat C_p and the constant-volume specific heat C_v . In a certain time interval Δt , the relationship formula between the air volume $\Psi_{air}(t)$ and the air volume in the next certain time interval $\Psi_{air}(t + \Delta t)$ is

$$\begin{aligned}
\Psi_{air}(t + \Delta t) &= \Psi_{air}(t) + \Delta \Psi_{air}(t + \Delta t) \\
&= \Psi_{air}(t) + A_e v_e(t) \Delta t
\end{aligned}$$
(14)

(4) Mass Change in Water Rocket

The amount of water lost $dm_i(t)$ in a certain time interval Δt after the launch of the water rocket is

$$dm_i(t) = \rho_{water} v_e(t) A_e \Delta t \tag{15}$$

where ρ_{water} is the density of water. The total mass of the water rocket at the time *t* is

$$M(t) = W + m_i(t) \tag{16}$$

where *W* is the mass of the water rocket shell. Therefore, the total mass of the water rocket at the time $t + \Delta t$ is

$$M(t + \Delta t) = W + m_i(t + \Delta t) = W + m_i(t) - dm_i(t)$$

= W + m_i(t) - \rho_{water} v_e(t) A_e \Delta t (17)

(5) Motion Equation of Water Rocket

In a certain time interval Δt , the relationship between the flight velocity of the water rocket V_{cv} and the flight acceleration of the water rocket \dot{V}_{cv} is

$$V_{cv}(t + \Delta t) = V_{cv}(t) + \dot{V}_{cv}(t)\Delta t$$
(18)

Thus, the flight altitude of the water rocket at the time $t + \Delta t$ is

$$S(t + \Delta t) = S(t) + V_{cv}(t)\Delta t$$
⁽¹⁹⁾

(6) Procedure for a Water Rocket's Flight Simulation

According to the above-mentioned momentum conservation equation, motion equation, and related formulas, under certain conditions, the propulsion performance and flight trajectory of a water rocket can be simulated using FORTRAN language through program codes to calculate the vertical flight distance, velocity, and acceleration. The calculation procedure is as follows:

For the water propulsion process, first enter the air pressure value within the water rocket P_i . The outlet pressure of the water rocket P_e is 1 atm (101,325 Pa). Thus, the initial value of the relative average velocity of fluid at the outlet v_e can be obtained from Eq. (11). Substitute v_e into Eq. (14) to find the air volume in the rocket at the next instant $V_{air}(t + \Delta t)$. Then, substitute $V_{air}(t + \Delta t)$ into Eq. (13) for the air pressure in the rocket at the next instant $P_i(t + \Delta t)$, and substitute $P_i(t + \Delta t)$ into Eq. (11) to find the outlet velocity at the next instant

 $v_{e}(t + \Delta t)$. The relative average acceleration of the fluid at the outlet $\dot{v}_e(t)$ is then obtained by means of Eq. (12). In terms of the mass of the water rocket and the mass change of the water within the water rocket, the value of $v_{e}(t)$ is put into the equation (15), and the mass of water in the water rocket $m_i(t)$ and the total mass of the water rocket M(t)at the time t are obtained by means of the equations (16) and (17), respectively. With the values of $v_{e}(t)$, $m_{i}(t)$, and M(t), Eq. (18) is substituted into Eq. (9) to obtain the acceleration of the water rocket $V_{cv}(t)$. Then, the flight velocity of the water rocket $V_{cv}(t + \Delta t)$ and the vertical flight altitude of the water rocket $S(t + \Delta t)$ in the next time interval Δt are obtained by Eqs. (18) and (19), respectively. The value of $v_{e}(t)$ derived from the previous time interval Δt is put into Eq. (14) to find the volume of the air within the water rocket at the time $t + \Delta t$, $V_{air}(t + \Delta t)$. According to the above steps, the velocity, acceleration, and flight altitude of the water rocket in the next time interval $V_{cv}(t + \Delta t)$, $V_{cv}(t)$, and $S(t + \Delta t)$ can be obtained. Follow this procedure to calculate until all the water in the water rocket is discharged.

For the air propulsion process, the air inside the water rocket is discharged like the water propulsion; however, P_eA_e can be neglected, the effect of the momentum change rate acting on the control volume is insignificant, and the momentum change rate acting on the control surface area can be changed to $-\rho_{air}v_e^2A_e$. Therefore, in the air propulsion process, Eq. (9) can be revised as

$$-\frac{1}{2}\rho_{air}V_{cv}^2AC_D - M(t)g = M(t)\dot{V}_{cv} - \rho_{air}v_e^2A_e \quad (20)$$

According to the above steps of the water propulsion process, the velocity, acceleration, and flight altitude of the water rocket in the next time interval $V_{cv}(t + \Delta t)$, $\dot{V}_{cv}(t)$, and $S(t + \Delta t)$ can also be obtained.

For free flight, there is no air flow out, that is $v_e = 0$. During this process, the air pressure in the water rocket P_i is equal to the air pressure outside the water rocket P_e . Therefore, in the free flight process, Eq. (9) can be revised as

$$-\frac{1}{2}\rho_{air}V_{cv}^2AC_D - M(t)g = M(t)\dot{V}_{cv}$$
(21)

According to the same steps mentioned above, the velocity, acceleration, and flight altitude of the water

rocket in the next time interval $V_{cv}(t + \Delta t)$, $V_{cv}(t)$, and $S(t + \Delta t)$ during free flight can also be obtained.

Flight Simulation Procedures

The flight process of a two-stage water rocket can be divided into two parts: two-stage combined flight and two-stage separated flight. The flight simulation procedures are as follows.

When the pressure difference between the second stage and the first stage of the water rocket is greater than the pressure difference that the one-way check valve can withstand, the second and first stages will fly together; when the pressure difference between the two stages is less than the pressure difference that the valve can withstand, the second stage will fly alone. For two-stage combined flight, water and high-pressure air provided in the first stage (lower part) of the water rocket are used as fuel to perform the water propulsion, air propulsion, and free flight processes. The flight simulation procedure can be regarded as the first stage flight simulation procedure, so as to determine V_{cv} , \dot{V}_{cv} , and S during the two-stage combined flight processes. As flight, for two-stage separated water and high-pressure air provided in the second stage (upper part) of the water rocket are used as fuel to perform the water propulsion, air propulsion, and free flight processes. The flight simulation procedure can be regarded as the second stage flight simulation procedure, so as to determine V_{cv} , \dot{V}_{cv} , and S during the two-stage separated flight processes.



Fig. 2. Experimental device for pressure difference test.

EXPERIMENTAL SETUP

Pressure Differences Test

Figure 2 shows the experimental device used to test the pressure difference resulted from the one-way check valve. In the experiment, an iron ring was used to fix the first stage of the water rocket to the iron frame, the check valve was installed at the bottle mouth above the first stage of the water rocket, and the second stage of the water rocket was connected above the check valve. During the pressure difference test, 1800 ml and 120 ml of water were first put into the first and second stages of the water rocket, respectively. Then, a pressurizing device was used to pressurize the air in the water rocket to 90 psi. Since the air entering the second stage of the water rocket through the one-way check valve from the first stage of the water rocket will not flow back into the first stage of the water rocket, the air pressures in the first and second stages of the water rocket are both 90 psi. Then, the launcher installed at the bottom of the first stage of the water rocket was pressed to discharge the water in the first stage until the second stage is separated from the first stage. The amount of water discharged was measured, the air volume in the first stage at the moment when the second stage was separated from the first stage was calculated, and the air pressure in the first stage can then be calculated under the isentropic process by using Eq. (13). By calculating the pressure difference between the second stage and the first stage of the water rocket, the pressure difference that the one-way check valve can withstand can be finally obtained. Here, in order to design a one-way check valve that can withstand different pressure differences, a certain number and size of O-rings (inner diameter 9 mm, thickness 2.6 mm) are fixed on the outer tube of the one-way check valve. Due to the different number of O-rings used, the binding force on the outer tube of the one-way check valve can be changed to control the pressure difference.

Flight Dynamic Test System

This experimental study is to measure the vertical flight height of a two-stage water rocket under specific experimental parameters, and compare the data obtained with the theoretical results. The height measurement device is a thin wire reel device, which is a device with a wire reel next to the two-stage water rocket It uses a thin wire to hook the tail of the second stage to ensure that the thin wire can be pulled up when the water rocket lifts off. As long as the vertical length of the elongated thin wire is measured, the vertical flight height of the water rocket can be obtained.

Before the start of the experiment, the pressure

difference (DP) test of the one-way check valve was performed. The test result was DP=3.75 atm. The working temperature was set to 25°C, the one-way check valve was installed in the two-stage water rocket, 120 ml and 600 ml of water were respectively filled in the second and first stages of the water rocket, and the air pressure in the water rocket was fixed at 8 atm. Other relevant experimental parameters were set as follows: $\rho_{air} = 1.1839 \text{ kg/m}^3$, $\rho_{water} = 997 \text{ kg/m}^3$, $C_D = 0.55$, $A_{il} = 0.00567 \text{ m}^2$ (the subscript 1 represents the first stage), A_{i2} = 0.00363 m^2 (the subscript 2 represents the second stage), A_{csl} = 0.00571 m^2 , $A_{cs2}=0.00369 \text{ m}^2$, $A_{el}=0.00028 \text{ m}^2$, $A_{e2}=$ 0.000012 m², P_{e} = 101,325 N/m² (1 atm), P_{il} = 810,400 N/m² (8 atm), P_{i2} =810,400 N/m² (8 atm), $\Psi_{wl} = 0.0006 \text{ m}^3$ (the subscript w represents water), V_{w2} = 0.00012 m³, W_I = 0.195 kg, W_2 = 0.174 kg, V_I = 0.0025 m³, $\Psi_2 = 0.0006$ m³, and $\Delta t = 0.0005$ s.

RESULTS AND DISCUSSION

Validation of Theoretical Model

According to the experimental parameters listed in the previous section, the experimental data and theoretical results of the vertical flight height of the water rocket were carried out and compared, as shown in Table 1. It can be seen from Table 1 that the absolute percentage error between the theoretical and experimental values in this study is only 3.96% to 5.65%, which shows that the model constructed by the theory of this study is quite reasonable.

Table 1. Comparison of theoretical and experimental results.

Experimental Height	Theoretical Height	Absolute Percentage
(m)	(m)	Error (%)
76.67±0.62	80.35	3.96 - 5.65



Fig. 3. Flight height versus *DP* for different values of P_i with $\Psi_{wl} = 0.0008 \text{ m}^3$ and $\Psi_{w2} = 0.0001 \text{ m}^3$.



Fig. 4. Flight height versus *DP* for different values of Ψ_{w1} with $P_{i1}=P_{i2}=8$ atm and $\Psi_{w2}=0.0001$ m³.

Pressure Difference Effect

Figure 3 shows the relationship between the flight height S and the pressure difference DP under different air pressures of $P_{i1}=P_{i2}=6$ atm, 7 atm, and 8 atm with the condition of the water amounts Ψ_{wl} = 0.0008 m³ and $\Psi_{w2} = 0.0001$ m³. The numerical results show that under the same water amount condition and the same air pressure, the greater the pressure difference that the one-way check valve can withstand, the higher the flight height of the two-stage water rocket. When the pressure inside the two-stage water rocket is greater, the one-way check valve needs to withstand a greater pressure difference so that the water in the first stage can be completely discharged to reach a higher flight height of the second stage. The results also show that the greater the pressure difference that the one-way check valve can withstand, the greater the water amount that the first stage can discharge before the stage is separated. As a result, the first stage can provide more momentum for the water rocket, so that the second stage can fly to a higher height after being separated.

Figure 4 shows the relationship between the flight height S and the pressure difference DP under different first-stage water amounts of $\Psi_{wl} = 0.0004 \text{ m}^3$, 0.0006 m^3 , and 0.0008 m^3 with the condition of the air pressure $P_{il}=P_{i2}=8$ atm and the water amount $\Psi_{w2}=0.0001$ m³. The numerical results show that under the same air pressure in the water rocket and the same water amount in the second stage, the more water the water rocket has in the first stage, the greater the pressure difference that the one-way check valve must withstand, so that the water in the first stage will be completely discharged before the stage is separated, and the second stage of the water rocket will fly to a higher flight height. On the contrary, under the same air pressure in the water rocket and the same water amount in the second stage, if there is more water in the first stage, it is more difficult to completely discharge the water before the first stage

is separated. As a result, if the one-way check valve cannot withstand enough pressure difference, the flight height of the second stage may be lower.



Fig. 5. Flight height versus *t* for different values of P_i with $\Psi_{wl} = 0.0008 \text{ m}^3$ and $\Psi_{w2} = 0.0001 \text{ m}^3$.



Fig. 6. Flight velocity versus *t* for different values of P_i with $\Psi_{w1} = 0.0008 \text{ m}^3$ and $\Psi_{w2} = 0.0001 \text{ m}^3$.



Fig. 7. Flight acceleration versus *t* for different values of P_i with $\Psi_{wl} = 0.0008 \text{ m}^3$ and $\Psi_{w2} = 0.0001 \text{ m}^3$.

Air Pressure Effect

Now we pay attention to the influence of the air pressure within the water rocket on the flight trajectory and dynamic characteristics of a two-stage water rocket theoretically. The relationship between the flight height S, velocity V_{cv} , and acceleration \dot{V}_{cv} and the time t under different air pressures of $P_{i1}=P_{i2}=6$ atm, 7 atm, and 8 atm based on the case of the water amounts $\Psi_{wl} = 0.0008 \text{ m}^3$ and $\Psi_{w2} = 0.0001$ m³ is presented in Figs. 5–7. The numerical results show that as the air pressure within the two-stage water rocket increases, the flight height increases, and the maximum flight velocity and maximum flight acceleration also increase. This means that when the air pressure within the water rocket is higher, the total momentum of the water rocket increases due to the increase in the outlet flow. Furthermore, the water rocket can reach a higher flight height, a faster flight velocity, and a greater flight acceleration.



Fig. 8. Flight height versus *t* for different values of V_{w1} with $P_{i1}=P_{i2}=8$ atm and $V_{w1}+V_{w2}=0.0009$ m³.



Fig. 9. Flight velocity versus *t* for different values of Ψ_{w1} with $P_{i1}=P_{i2}=8$ atm and $\Psi_{w1}+\Psi_{w2}=0.0009$ m³.



Fig. 10. Flight acceleration versus *t* for different values of V_{w1} with $P_{i1}=P_{i2}=8$ atm and $V_{w1}+V_{w2}=0.0009$ m³.

First-Stage Water Amount Effect

Now we pay attention to the influence of the water amount within the first stage (lower part) of the water rocket on the flight trajectory and dynamic characteristics of a two-stage water rocket theoretically. The relationship between the flight height S, velocity V_{cv} , and acceleration \dot{V}_{cv} and the time t under different first-stage water amount of $V_{wl}=0.0005 \text{ m}^3$, 0.00065 m³, and 0.0008 m³ based on the case of the air pressure $P_{i1}=P_{i2}=8$ atm and total water amount $V_{w1} + V_{w2} = 0.0009 \text{ m}^3$ is presented in Figs. 8-10. The numerical results show that as the water amount in the first stage increases from 0.0005 m^3 to 0.00065 m^3 (the water amount in the second stage of the water rocket decreases from 0.0004 m³ to 0.00025 m^3), the flight height of the water rocket S increases, and the maximum flight velocity V_{cv} and maximum flight acceleration V_{cv} also increase. When the water amount in the second stage of the water rocket continues to increase from 0.00065 m³ to 0.0008 m³ (the water amount in the second stage of the water rocket decreases from 0.00025 m³ to 0.0001 m³), the flight duration (the length of time the water rocket can stay in the air), maximum flight velocity V_{cv} and maximum flight acceleration V_{cv} continue to increase, while the flight height of the water rocket S decreases; that is, an optimal flight height is reached for a fixed total water amount in the two-stage water rocket. This means that if the water amount in the lower part of the water rocket changes from less to more (the water amount in the upper part of the water rocket changes from more to less), the flight duration, flight velocity, and acceleration could increase, but the height to be pushed could be reduced. Therefore, through the adjustment of the first-stage water amount fraction, the propulsion effect may be maximized.

CONCLUSIONS

This study has completed the exploration of the flight trajectory and dynamic characteristics of a two-stage water rocket. In the process of establishing the mathematical model, the Reynolds transportation theorem was first used to derive the momentum conservation equation of a water rocket. Then, in the processes of water propulsion, air propulsion, and free flight, the continuity equation, the momentum conservation equation, the Bernoulli equation, and the isentropic process relationship were applied to establish the theoretical flight model of the water rocket. Finally, a two-stage water rocket numerical simulation was carried out for this flight mode and compared with the experimental data. The purpose of the study is to investigate the effects of pressure difference, air pressure, first-stage water amount on the vertical flight height, velocity, and acceleration of the water rocket over time. The main results obtained are summarized as follows:

- 1. The greater the pressure difference that the one-way check valve can withstand, the higher the height after being separated.
- 2. As the air pressure in the two-stage water rocket increases, the flight height of the water rocket increases, and the maximum flight velocity and maximum flight acceleration also increase.
- 3. If the water amount in the first stage of the water rocket changes from less to more, the flight duration, flight velocity, and acceleration could increase, but the height to be pushed could be reduced. Through the adjustment of the first-stage water amount fraction, the propulsion effect may be maximized.

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雙節式水火箭飛行動態特 性分析

張峰壵 翁輝竹 中原大學機械工程學系

摘要

本研究以理論模式探討雙節式水火箭的飛行 軌跡及動力學特性。主要目的是分析水火箭第一節 與第二節中氣壓及水量隨時間對第二節(水火箭上 部)的垂直飛行高度、速度及加速度之影響。飛行 過程中,第一節和第二節之間的單向逆止閥所引起 的壓差是決定兩節分離的關鍵因素。該飛行模型係 基於水火箭動量守恆方程、出口流體速度條件,以 及內部氣壓與水量之間的關係。透過對該問題的數 值求解,可預測水火箭的垂直飛行高度、速度及加 速度。首先將所預測之結果與實驗數據進行比較, 藉以獲得良好驗證。進一步研究發現,雙節式水火 箭中的逆止閥可有效地使火箭的垂直飛行高度高 於單節式水火箭的垂直飛行高度,而單向逆止閥可 承受的壓差越大,從第一節分離後第二節持續飛行 的高度將越高。在特定氣壓及水量的情況下,藉由 調節水火箭上部和下部之間的水量體積比可實現 最佳的飛行性能。