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Solutions for Unsteady Unidirectional Flow of a Generalized Burger Fluid in an Annular Pipe With Different Given Volume Flow Rate Conditions

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ABSTRACT

The velocity profile and pressure gradient of the unsteady state unidirectional flow of a generalized Burger fluid in an annular pipe with different given volume flow rate conditions are investigated in this study. Traditional models of this fluid is often solved by a partial (or ordinary) differential equation (PDE/ODE) with initial values or some boundary conditions. However, it is not enough to describe the phenomenon of Burger fluid in the real case. In this research, we added the inlet volume flow rate as an initial condition of the PDE. In order to understand the different flow characteristics, two basic flow situations are solved based on the prescribed flow conditions including a suddenly started flow and a constant accelerated flow, respectively. Finally, the linear acceleration and oscillatory flow is also considered in the last two cases.

INTRODUCTION

Scientist and engineers are interested in the Newtonian fluid because it can be used in a wide range of applications, such as chemical engineering, mechanical engineering, petroleum engineering, nuclear industries, geophysics and bioengineering. Burger fluid is one kind of the Newtonian fluids. This fluid is quite special because its' partial differential equation (PDE) is linear and it is more complicated than Oldroyd-B fluid. Many researchers focused on the study of Burger fluid and its qualitative properties.

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For example, Hu et al. (2013) got unstable modes of Burger fluid. Some investigators studied the numerical solution related to Burger fluid. For instance, Tripathi (2011) focused on the numerical study of peristaltic flow of generalized Burger fluid. Recently, great attention had been paid to the exact solution (Tong and Shan 2009, Hayat et al. 2011, Khan et al. 2011). Hayat et al. (2011) developed the exact solution of a non-Newtonian fluid between the micro-parallel plates. Khan et al. (2011) got the exact solutions of Stokes second problem. Besides the exact solutions of some unsteady lows of generalized Burgers' fluid in an annular pipe are obtained by Tong and Shan (2009).

The present study suggests inlet volume flow rate as a condition of the PDE. In traditional, a model of the fluid often be a PDE with initial values or some boundary values. In real world, initial values and boundary values are not enough to describe the phenomenon of Burger fluid. Add the inlet volume flow rate is more actually than traditional conditions. In this paper, we have solved the governing equation with initial, boundary, inlet volume flow rate conditions by Laplace transform and the exact solution is obtained. The flow behavior could be explained by these analytical solutions in the following cases.

METHODOLOGY

Mathematical formulations

The assumptions of the unsteady flow of the incompressible fluid of a Burger type in an annular pipe are as follows:

(1) The fluid velocity of the direction of the pipe radius is zero.
(2) The flows are axisymmetric pipe-like.
(3) The axial velocity is only dependent on the pipe radius.

The generalized Burger fluid has the form with the constitutive relationship of t as follows (Khan et al. 2011):

$$(1+\lambda_1\partial_t+\lambda_2\partial_t^2)\tau=\mu(1+\lambda_3\partial_t+\lambda_4\partial_t^2)\partial_r u \qquad (2.1)$$

where τ is the tangential stress tension, λ_1 and λ_3 are the relaxation and retardation times, respectively, λ_2 and λ_4 are the new material parameters of the generalized Burger' fluid, μ is the dynamic viscosity and u is the velocity.

The motion equation of the axial flow is written as

$$\rho\partial_t w = \partial_r \tau + \frac{1}{r}\tau - \partial_z p \qquad (2.2)$$

where ρ is the constant density of the fluid.

Inserting (2.1) into (2.2) eliminating τ yields

$$\begin{aligned}&(1+\lambda\partial_t + k\partial_t^2)\partial_t u\\ &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + (\nu + l\partial_t + q\partial_t^2)(\partial_r^2 + \frac{1}{r}\partial_r)u(r,t)\end{aligned} \qquad (2.3)$$

where $-\frac{1}{\rho}\frac{\partial p}{\partial z}$ is the pressure gradient that acts on the liquid along the z-direction, $\nu = \frac{\mu}{\rho}$ is the kinematical viscosity and $\lambda = \lambda_1, k = \lambda_2, l = \nu\lambda_3, q = \nu\lambda_4$.

Methodology of solution

Since the governing equation, boundary conditions and initial condition are known, the problem is well posed. In general, it is not convenient to solve this equation by the method of separation of variables or eigenfunction expansion. In this study, the Laplace transform method is used to reduce two variables into single variable. This procedure greatly reduces the difficulties of treating the original differential equation.

The governing equation of motion in z-direction is

$$\begin{aligned}&(1+\lambda\partial_t + k\partial_t^2)\partial_t u\\ &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + (\nu + l\partial_t + q\partial_t^2)(\partial_r^2 + \frac{1}{r}\partial_r)u(r,t).\end{aligned} \qquad (3.1)$$

As the radius of duct is R and the boundary conditions are

$$u(R_1,t) = 0\,, \qquad (3.2)$$

$$u(R_2,t) = 0\,. \qquad (3.3)$$

The problem can be solved if the pressure gradient is known. In this study, the pressure gradient is determined indirectly by

$$\int_{R_1}^{R_2} 2\pi r u(r,t)dr = \pi(R_2^2 - R_1^2)u_p(t) = Q(t) \qquad (3.4)$$

where $u_p(t)$ is the given average inlet velocity and Q(t) is known inlet volume flow rate. Eq. (3.4) is termed as additional condition.

The above governing equation, boundary conditions and initial condition are prescribed and can be solved by the Laplace transform technique, which yields the following equations:

governing equation

$$\begin{aligned}&\frac{\partial^2 u(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,s)}{\partial r} - \frac{s+\lambda s^2 + ks^3}{\nu + ls + qs^2}u(r,s)\\ &= \frac{1}{\rho}\frac{1}{(\nu + ls + qs^2)}\frac{\partial p(z,s)}{\partial z}\end{aligned} \qquad (3.5)$$

with boundary conditions

$$u(R_1,s) = 0 \qquad (3.6)$$

$$u(R_2,s) = 0 \qquad (3.7)$$

and additional condition

$$\int_{R_1}^{R_2} 2\pi r u(r,s)dr = \pi(R_2^2 - R_1^2)u_p(s)\,. \qquad (3.8)$$

Eq. (3.5) is a second order non-homogeneous ordinary differential equation. The homogeneous part is the modified Bessel's equation of zeroth order and the particular solution is assumed to be Ψ, the complete solution is

$$u(r,s) = aI_0(mr) + bK_0(mr) + \Psi \qquad (3.9)$$

where $m = (\frac{s+\lambda s^2 + ks^3}{\nu + ls + qs^2})^{\frac{1}{2}}$.

The boundary conditions Eq. (3.6) and Eq. (3.7) are used to solve for the two arbitrary coefficients a and b. Substituting Eq. (3.6) and Eq. (3.7) into Eq. (3.9) gives

$$\begin{aligned}&b = -\frac{T}{H}a,\\ &\Psi = a[\frac{T}{H}K_0(mR_1) - I_0(mR_1)]\end{aligned} \qquad (3.10)$$

where $H = K_0(mR_2) - K_0(mR_1), T = I_0(mR_2) - I_0(mR_1)$.

From the additional condition of Eq. (3.8),

$$\begin{aligned}&\int_0^R 2\pi r[aI_0(mr) + bK_0(mr) + \Psi]dr\\ &= u_p(s)\pi(R_2^2 - R_1^2)\end{aligned} \qquad (3.11)$$

substituting Eq. (3.10), a is readily obtained as

$$a = \frac{u_p(s)(R_2^2 - R_1^2)}{2PH - 2VT + [mTK_0(mR_1) - mHI_0(mR_1)](R_2^2 - R_1^2} \qquad (3.12)$$

, $V = R_1K_1(mR_1) - R_2K_1(mR_2), P = R_2I_1(mR_2) - R_1I_1(mR_1).$

Substituting Eq. (3.10) into Eq. (3.9), gives

$$u(r,s) = \frac{a}{H}\{H[I_0(mr) - I_0(mR_1)] - T[K_0(mr) - K_0(mR_1 \qquad (3.13)$$

or

$$u(r,s) = u_p(s)\Omega(r,s) \qquad (3.14)$$

where

$$\begin{aligned}&\Omega(r,s)\\ &= \frac{A}{H}\{H[I_0(mr) - I_0(mR_1)] - T[K_0(mr) - K_0(mR_1)]\end{aligned} \qquad (3.15)$$

where $A = \frac{(R_2^2 - R_1^2)}{2PH - 2VT + [mTK_0(mR_1) - mHI_0(mR_1)](R_2^2 - R_1^2)}$.

Taking the inverse Laplace transform, the velocity profile is

$$u(r,t) = \frac{1}{2\pi i}\int_{\nu - i\infty}^{\nu + i\infty} u_p(s)\Omega(r,s)e^{st}ds\,. \qquad (3.16)$$

Furthermore, the pressure gradient is found by substituting Eq. (3.10) into Eq. (3.5) to give

$$\frac{\partial p(z,s)}{\partial z} = u_p(s)(-\rho)(s+\lambda s^2+ks^3)\Psi \tag{3.17}$$

or

$$\frac{\partial p(z,s)}{\partial z} = u_p(s)(-\rho)(s+\lambda s^2+ks^3)A[\frac{T}{H}K_0(mR_1)-I_0(mR_1)] \tag{3.18}$$

Using the inverse transform formula, the pressure gradient distribution can also be obtained.

ILLUSTRATION OF CASES

In this study, we solved different cases to understand the different flow characteristics. In first case, the piston velocity $u_p(t)$ moves with a constant acceleration. The piston starts suddenly from the rest and then maintains this velocity in the second case. Finally, the linear acceleration and oscillatory piston motion are also considered in the last two cases, respectively.

Case 1. Constant acceleration piston motion

The piston motion of constant acceleration can be described by the following equation.

$$u_p(t) = a_p t = \left(\frac{U_p}{t_0}\right)t \tag{4.1}$$

where a_p is the constant acceleration, U_p is the final velocity after acceleration, and t_0 is the time period of acceleration.

Taking the Laplace transform of Eq. (4.1) and then get

$$u_p(s) = \frac{U_p}{t_0 s^2}\cdot \tag{4.2}$$

From Eq. (3.16), we obtain the velocity profile as

$$u(r,s) = \frac{U_p}{2\pi t_0 i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{A}{s^2H}\{H[I_0(mr)-I_0(mR_1)]-T[K_0(mr)-K_0(mR_1)]e^{st}ds\} \tag{4.3}$$

From the above expression, the integration is determined using complex variable theory, as discussed by Arpaci [6]. It is easily observed that $s=0$ is a pole of order 2. Therefore, the residue at $s=0$ is

$$\mathrm{Res}(0) = \frac{U_p}{t_0}[(ar^2+c+d\ln r)t+\frac{a}{16\nu}r^4+\frac{d}{4\nu}r^2\ln r+gr^2+hh]$$

where

$$d=\frac{2(R_2^2-R_1^2)}{P}, a=\frac{2}{P}\ln\frac{R_2}{R_1}, c=-aR_1^2-d\ln R_1, \tag{4.4}$$

$$P=(R_1^2-R_2^2)(\ln R_1R_2+1)+2R_2^2\ln R_2-2R_1^2\ln R_1,$$

$$g=\frac{1}{(R_1^2-R_2^2)}[\frac{a}{16\nu}(R_2^4-R_1^4)+\frac{d}{4\nu}(R_2^2\ln R_2-R_1^2\ln R_1)],$$

$$hh=\frac{1}{(R_1^2-R_2^2)}[\frac{a}{48\nu}(R_2^6-R_1^6) +(\frac{g}{2}-\frac{d}{64\nu})(R_2^4-R_1^4) +\frac{d}{16\nu}(R_2^4\ln R_2-R_1^4\ln R_1)].$$

The other singular points are the roots of following transcendental equation:

$$H\times\{2PH-2VT+[mTK_0(mR_1)-mHI_0(mR_1)](R_2^2-R_1^2) \tag{4.5}$$

Setting $m=i\alpha$, we have

$$H_\alpha\times\{2P_\alpha H_\alpha-2V_\alpha T_\alpha+[\alpha T_\alpha Y_0(\alpha R_1)-\alpha H_\alpha J_0(m\alpha R_1)](R_2^2-R_1^2)=0$$
$$I_0(mR_1)=J_0(\alpha R_1), K_0(mR_1)=Y_0(\alpha R_1), \tag{4.6}$$

$$H_\alpha=Y_0(\alpha R_2)-Y_0(\alpha R_1), T=J_0(\alpha R_2)-J_0(\alpha R_1),$$
$$V_\alpha=R_1Y_1(\alpha R_1)-R_2Y_1(\alpha R_2), P_\alpha=R_2J_1(\alpha R_2)-R_1J_1(\alpha R_1). \tag{4.7}$$

If α_n, $n=1,2,3,...,\infty$, are zeros of Eq. (4.6), then s_{1n}, s_{2n}, s_{3n} are roots of

$$ks^3+(\lambda+\alpha_n^2q)s^2+(1+\alpha_n^2 l)s+\alpha_n^2\nu=0, \tag{4.8}$$

$n=1,2,3,...,\infty$. These are simple poles, and the residues at all of these poles can be obtained as

$$\mathrm{Res}(s_{in})=\frac{U_p}{t_0}\frac{e^{s_{i1n}t}RR(r)}{s_{in}^2Q'(s_{in})},\ i=1,2,3 \tag{4.9}$$

where

$$RR(r)=(R_2^2-R_1^2)\{H[I_0(mr)-I_0(mR_1)]-T[K_0(mr)-K_0(mR_1)]\},$$
$$Q'(s_{in})=H'_\alpha W+H_\alpha W', i=1,2,3., \tag{4.10}$$

where

$$W=2P_\alpha H_\alpha-2V_\alpha T_\alpha+[\alpha T_\alpha Y_0(\alpha R_1)-\alpha H_\alpha J_0(m\alpha R_1)](R_2^2-R_1^2),$$
$$W'=\frac{\partial\alpha}{\partial s}[2P'_\alpha H_\alpha+2P_\alpha H'_\alpha-2V'_\alpha T_\alpha-2V_\alpha T'_\alpha+[T_\alpha Y_0(\alpha R_1)+\alpha T'_\alpha Y_0(\alpha R_1)$$
$$-R_1\alpha T_\alpha Y_1(\alpha R_1)-H_\alpha J_0(\alpha R_1)-\alpha H'_\alpha J_0(\alpha R_1)-R_1\alpha H_\alpha J_1(\alpha R_1)](R_2^2-R_1^2) \tag{4.11}$$
$$H'_{i\alpha}=\frac{\partial\alpha_{in}}{\partial s}[R_2Y_1(\alpha R_2)-R_1Y_1(\alpha R_1)],$$

$$\frac{\partial\alpha}{\partial s}=-\frac{1}{2\alpha}\frac{(\nu+ls+qs^2)(1+2\lambda s+3ks^2)-(s+\lambda s^2+ks^3)(l+2qs)}{(\nu+ls+qs^2)^2}. \tag{4.12}$$

Adding $\mathrm{Res}(0)$, $\mathrm{Res}(s_{1n})$, $\mathrm{Res}(s_{2n})$ and $\mathrm{Res}(s_{3n})$ a complete solution for constant acceleration case is obtained as

$$\frac{u(r,t)t_0}{U_p}=(ar^2+c+d\ln r)t+\frac{a}{16\nu}r^4+\frac{d}{4\nu}r^2\ln r+gr^2 +hh+\sum_{n=1}^{\infty}\left(\frac{e^{s_{1n}t}}{s_{1n}{}^2Q'(s_{1n})}+\frac{e^{s_{2n}t}}{s_{2n}{}^2Q'(s_{2n})}+\frac{e^{s_{3n}t}}{s_{3n}{}^2Q'(s_{3n})}\right)RR(r), \tag{4.13}$$

where $RR(r)$, $Q'(s_{in}), i=1,2,3$, are defined in Eqs. (4.9)-(4.12).

The first term on the right-hand side of Eq. (4.13) represents the steady state velocity, the second term is the transient response of the flow to an abrupt change either in the boundary conditions, body forces, pressure gradient or other external driving force.

Eq. (3.20) is used to determine the pressure gradient in this flow field, and follows the same procedure for solving velocity profile

$$\mathrm{Res}(0)=\frac{\rho U_p}{t_0}(-t\frac{8\nu}{P}\ln\frac{R_2}{R_1}-4\nu(f+g)+c-4al), \tag{4.14}$$

$$\mathrm{Res}(s_{1n}) = -\frac{\rho U_p}{t_0}\frac{PP(s_{1n})}{s_{1n}{}^2 Q'(s_{1n})}e^{s_{1n}t},$$
$$PP(s_{in}) = (s_{in} + \lambda s_{in}{}^2 + k s_{in}{}^3)[\frac{T}{H}K_0(mR_1) - I_0(mR_1)](R_2^2 - R_1^2), \tag{4.15}$$

i=1,2,3

$$\mathrm{Res}(s_{2n}) = -\frac{\rho U_p}{t_0}\frac{PP(s_{2n})}{s_{2n}{}^2 Q'(s_{2n})}e^{s_{2n}t},$$
$$\mathrm{Res}(s_{3n}) = -\frac{\rho U_p}{t_0}\frac{PP(s_{3n})}{s_{3n}{}^2 Q'(s_{3n})}e^{s_{3n}t}. \tag{4.16}$$

Therefore, the pressure gradient is

$$\frac{\partial p(z,t)}{\partial z} = \frac{\rho U_p}{t_0}\left\{t\frac{8\nu}{P}\ln\frac{R_2}{R_1} - 4\nu(f+g) + c - 4al \right.$$
$$\left. -\sum_{n=1}^{\infty}\left[\frac{PP(s_{1n})}{s_{1n}{}^2 Q'(s_{1n})}e^{s_{1n}t} + \frac{PP(s_{2n})}{s_{2n}{}^2 Q'(s_{2n})}e^{s_{2n}t} + \frac{PP(s_{3n})}{s_{3n}{}^2 Q'(s_{3n})}e^{s_{3n}t}\right]\right\} \tag{4.17}$$

where $PP(s_{in}), i = 1,2,3$ are defined in Eq. (4.15), $Q'(s_{in}), i = 1,2,3$, are defined in Eqs. (4.9)-(4.12).

Case 2. Suddenly started flow

For a suddenly started flow between the parallel surfaces, the piston motion can be described as follows

$$u_p = \begin{cases} 0 & \text{for } t \le 0, \\ U_p & \text{for } t > 0, \end{cases} \tag{4.18}$$

where U_p is the constant velocity.

In which case the velocity profile is

$$\frac{u(r,t)}{U_p} = ar^2 + c + d\ln r$$
$$+\sum_{n=1}^{\infty}\left(\frac{e^{s_{1n}t}}{s_{1n}Q'(s_{1n})} + \frac{e^{s_{2n}t}}{s_{2n}Q'(s_{2n})} + \frac{e^{s_{3n}t}}{s_{3n}Q'(s_{3n})}\right)RR \tag{4.19}$$

where

$$a = -\frac{2}{P}\ln\frac{R_2}{R_1}, d = \frac{a}{\ln\frac{R_2}{R_1}}(R_1^2 - R_2^2), c = -aR_1^2 - d\ln R_1,$$

$P = (R_1^2 - R_2^2)(\ln R_1R_2 + 1) + 2R_2^2\ln R_2 - 2R_1^2\ln R_1$,

$Q'(s_{in}), i = 1,2,3$ are defined in Eqs. (4.11) and (4.12), and the pressure gradient is

$$\frac{\partial p(z,t)}{\partial z} = \rho U_p\left\{\frac{8\nu}{P}\ln\frac{R_2}{R_1} - \sum_{n=1}^{\infty}\left[\frac{PP(s_{1n})}{s_{1n}Q'(s_{1n})}e^{s_{1n}t} + \frac{PP(s_{2n})}{s_{3n}Q'(s_{3n})}e^{s_{3n}t} + \frac{PP(s_{3n})}{s_{3n}Q'(s_{3n})}e^{s_{3n}t}\right.\right. \tag{4.20}$$

where $PP(s_{in}), i = 1,2,3$ are defined in Eq. (4.15), $RR(r)$, $Q'(s_{in}), i = 1,2,3$, are defined in Eqs. (4.09)-(4.12).

Case 3. Linear acceleration piston motion

The piston motion of linear acceleration can be described by the following equation:

$$u_p(t) = a_p t^2 = \left(\frac{U_p}{t_0}\right)t^2 \tag{4.21}$$

where a_p is the constant acceleration, U_p is the final velocity after acceleration, and t_0 is the time period of acceleration.

In which case the velocity profile is

$$\frac{u(r,t)t_0}{U_p} = (ar^2 + c + d\ln r)t^2 + [\frac{a}{8\nu}r^4 + \frac{d}{2\nu}r^2\ln r + kkr^2 + m]t$$
$$+ Jr^6 + Lr^4 + Mr^2 + N + \frac{d}{32\nu}r^4\ln r + \frac{d}{2}(\lambda - \frac{l}{\nu})r^2\ln r$$
$$+2\sum_{n=1}^{\infty}\left(\frac{e^{s_{1n}t}}{s_{1n}{}^3 Q'(s_{1n})} + \frac{e^{s_{2n}t}}{s_{2n}{}^3 Q'(s_{2n})} + \frac{e^{s_{3n}t}}{s_{3n}{}^3 Q'(s_{3n})}\right)RR(r) \tag{4.22}$$

where $d = \frac{2(R_2^2 - R_1^2)}{P}, a = \frac{2}{P}\ln\frac{R_2}{R_1}, c = -aR_1^2 - d\ln R_1$,

$$P = (R_1^2 - R_2^2)(\ln R_1R_2 + 1) + 2R_2^2\ln R_2 - 2R_1^2\ln R_1,$$
$$kk = \frac{1}{(R_1^2 - R_2^2)}[\frac{a}{8\nu}(R_2^4 - R_1^4) + \frac{d}{2\nu}(R_2^2\ln R_2 - R_1^2\ln R_1)],$$
$$m = \frac{1}{(R_1^2 - R_2^2)}[\frac{a}{24\nu}(R_2^6 - R_1^6) + (\frac{kk}{2} - \frac{d}{16\nu})(R_2^4 - R_1^4) + \frac{d}{4\nu}(R_2^4\ln R_2 - R_1^4\ln R_1)],$$
$$J = \frac{a}{288\nu^2},$$
$$L = \frac{1}{16\nu}(kk + 2\lambda a - 2\frac{al}{\nu} - \frac{d}{4\nu}),$$
$$M = \frac{1}{(R_1^2 - R_2^2)}[J(R_2^6 - R_1^6) + L(R_2^4 - R_1^4) + \frac{d}{32\nu^2}(R_2^4\ln R_2 - R_1^4\ln R_1)$$
$$+\frac{d}{2}(\lambda - \frac{l}{\nu})(R_2^2\ln R_2 - R_1^2\ln R_1)],$$
$$N = \frac{1}{(R_1^2 - R_2^2)}[\frac{J}{4}(R_2^8 - R_1^8) + (\frac{L}{3} - \frac{d}{576\nu^2})(R_2^6 - R_1^6) + [\frac{M}{2} - \frac{d}{16}(\lambda - \frac{l}{\nu})](R_2^4 - R_1^4)$$
$$+\frac{d}{96\nu^2}(R_2^6\ln R_2 - R_1^6\ln R_1) - \frac{d}{16}(\lambda - \frac{l}{\nu})(R_2^4\ln R_2 - R_1^4\ln R_1)].$$

and the pressure gradient is

$$\frac{\partial p(z,t)}{\partial z} = \frac{\rho}{t_0}U_p\left\{-t\frac{8\nu}{P}\ln\frac{R_2}{R_1} - d - 4\nu g + c - 4al\right.$$
$$\left.-\sum_{n=1}^{\infty}\left[\frac{PP(s_{1n})}{s_{1n}{}^3 Q'(s_{1n})}e^{s_{1n}t} + \frac{PP(s_{2n})}{s_{2n}{}^3 Q'(s_{2n})}e^{s_{2n}t} + \frac{PP(s_{3n})}{s_{3n}{}^3 Q'(s_{3n})}e^{s_{3n}t}\right]\right\} \tag{4.23}$$

where $PP(s_{in}), i = 1,2,3$ are defined in Eq. (4.15), $RR(r)$, $Q'(s_{in}), i = 1,2,3$, are defined in Eqs. (4.9)-(4.12).

Case 4. Oscillatory piston motion

The oscillating piston motion starting from rest is considered. The piston motion is described as

$$u_p = \begin{cases} 0 & \text{for } t \le 0, \\ U_0\sin(\omega t) & \text{for } t > 0, \end{cases} \tag{4.24}$$

Taking the Laplace transform of Eq. (4.24), and we have

$$\hat{u}_p(s) = \frac{U_0\omega}{s^2 + \omega^2}\ s > 0\,. \tag{4.25}$$

Substituting Eq. (4.25) into Eq. (3.18) to find the velocity profile. The poles are simple poles at $s = \pm i\omega$ and the roots of $\alpha h\cos\alpha h - \sin\alpha h = 0$. The solution to the velocity profile is

$$\frac{u(r,t)}{U_0} = \frac{i}{2}\left[e^{-i\omega t}\Omega(y,-i\omega) - e^{i\omega t}\Omega(y,i\omega)\right]$$
$$+\sum_{n=1}^{\infty}\left(\frac{e^{s_{1n}t}}{(s_{1n}{}^2 + \omega^2)Q'(s_{1n})} + \frac{e^{s_{2n}t}}{(s_{2n}{}^2 + \omega^2)Q'(s_{2n})} + \frac{e^{s_{3n}t}}{(s_{3n}{}^2 + \omega^2)Q'(s_{3n})}\right)RR \tag{4.26}$$

where $\Omega(y,s)$ is defined by Eq. (3.17), and the pressure gradient is obtained as

$$\frac{\partial p(z,t)}{\partial z}=-\frac{\rho U_0}{2}\left\{\begin{array}{l}\left(i\rho\omega+\sigma B_0{}^2\right)e^{i\omega t}\Gamma(i\omega)\\+\left(-i\omega\rho+\sigma B_0{}^2\right)e^{-ii\omega t}\Gamma(-i\varpi)\\+\sum_{n=1}^{\infty}\left[\begin{array}{l}\frac{PP(s_{1n})}{\left(s_{1n}{}^2+\omega^2\right)Q'(s_{1n})}e^{s_{1n}t}\\+\frac{PP(s_{2n})}{\left(s_{2n}{}^2+\omega^2\right)Q'(s_{3n})}e^{s_{2n}t}\\+\frac{PP(s_{3n})}{\left(s_{3n}{}^2+\omega^2\right)Q'(s_{3n})}e^{s_{3n}t}\end{array}\right]\end{array}\right\} \quad (4.27)$$

where $PP(s_{in}), i=1,2,3$ are defined in Eq. (4.15), $RR(r)$, $Q'(s_{in}), i=1,2,3$, are defined in Eqs. (4.9)-(4.12) and

$$\Gamma(s)=(s+\lambda s^2+ks^3)A[\frac{T}{H}K_0(mR_1)-I_0(mR_1)]$$

and $m=(\dfrac{s+\lambda s^2+ks^3}{\nu+ls+qs^2})^{\frac{1}{2}}$. (4.28)

CONCLUSIONS

The analytical solutions of velocity profile and pressure gradient to the different types of piston motion that corresponding to the given inlet volume flow rate are solved by using Laplace transform. The pressure gradient for each flow condition can be derived from the given function of volume flow rate with the same method. It is very beautiful for fully developed flows that the relaxation time only appears as the motion is unsteady.

The present method only can be used for models of linear PDE due to the limitation of Laplace transform. However, it still remains open and needs further investigation for models of nonlinear PDE in the future.

CONFLICTS OF INTEREST

The authors declare that there is no actual or potential conflict of interest regarding the publication of this article.

REFERENCES

Arpaci, V. S., *Conduction Heat Transfer*, Addison-Wesley Publishing Company, USA (1996).

Hayat, T., Afzal, S., and Hendi, A., "Exact solution of electroosmotic flow in generalized Burgers fluid," *Appl. Math. Mech.*, Vol. 32, Issue 9, pp. 1119-1126 (2011).

Hu, K.X., Peng, J., and Zhu, K.Q., "Linear stability of plane creeping Couette flow for Burgers fluid," *Acta Mech. Sin.*, Vol. 29, Issue 1, pp. 12-23 (2013).

Khan, M., Anjum, A., and Fetecau, C., "On exact solutions of Stokes second problem for a Burgers' fluid, II. The cases γ = λ2/4 and γ > λ2/4," *Z. Angew. Math. Phys.*, Vol. 62, Issue 4, pp. 749-759 (2011).

Tong, D., and Shan, L., "Exact solutions for generalized Burgers' fluid in an annular pipe," *Meccanica*, Vol. 44, Issue 4, pp. 427-431 (2009).

Tripathi, D., "Numerical study on peristaltic flow of generalized Burgers' fluids in uniform tubes in the presence of an endoscope," *Int. J. Numer. Method. Biomed. Eng.*, Vol. 27, Issue 11, pp. 1812-1828 (2011).

環形管道中的廣義 Burger 流體之非穩定單向流動在給定不同體積流量條件下之解

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摘 要

本研究在給定不同體積流量之條件下，探討其在環形管道中之廣義Burger流體的非穩態單向流動的速度分佈和壓力梯度。這種流體的傳統模型通常透過具有初始值或一些邊界條件的偏（或常）微分方程（PDE / ODE）來求解。然而，在實際情況下，用描述Burger流體的現象是不夠的。在這個研究中，我們增加了入口體積流量作為PDE的初始條件。為了了解不同的流動特性，在給定流動條件之下，分別求解包括突然開始流動和恆定加速流動的兩個基本流動情況。最後，也考慮了線性加速度和振盪流動之流動條件在最後的兩種情況。